FROBENII AND ITS ACTION ON PROJECTIVE SPACE

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Contents

Introduction and conventions

In these notes, we will introduce the notion of Frobenius and compute its action on étale cohomology of projective space. In doing so, we will state fundamental theorems of étale cohomology, introduce the notion of a twist, and finally utilize the action to count points on projective space.

Throughout these notes, we will assume that all varieties are irreducible, that $p > 0$ is a prime number, $q = p^f$ for some $f > 0$, and that $\Lambda = \underline{\mathbb{Z}/m\mathbb{Z}}$ for some natural number$m > 0$.

1. Preliminary remarks

We begin with a few preliminary remarks, to give meaning to the later sections.

Proposition 1.1. *Let $f\colon X \to Y$ be a map of schemes, and let $\mathcal{G} \in \mathbf{Sh}(Y_{\text{et}})$. Then there are natural maps*

$$f^*\colon H^i(Y_{\text{et}},\mathcal{G}) \to H^i(X_{\text{et}},f^*\mathcal{G})$$

If $g\colon Y \to Z$ is another map of schemes, then $(g \circ f)^* \simeq f^* \circ g^*$.

Proof. [Mil80, Remark III.1.6]. □

We note that.

Remark 1.2. We note that for $\pi\colon X \to Y$ we have that $\pi^*\mu_m \simeq \mu_m$ canonically, and for any constant sheaf $\mathcal{G} \in \mathbf{Sh}^{ab}(Y_{\text{et}})$, we have $\pi^*\mathcal{G} \simeq \mathcal{G}$. Thus f induces maps

$$\pi^*\colon H^i(Y_{\text{et}},\mu_l) \to H^i(X_{\text{et}},\mu_l)$$

$$\pi^*\colon H^i(Y_{\text{et}},\mathcal{G}) \to H^i(X_{\text{et}},\mathcal{G}).$$

We will also need the following.

Proposition 1.3. *If $f\colon X \to Y$ be a map of schemes, then the following diagram commutes*

$$\begin{tikzcd} \mathbf{Sh}(Y_{\text{Zar}}) \arrow[r, "\text{et}"] \arrow[d, "f^*"] & \mathbf{Sh}(X_{\text{et}}) \arrow[d, "f^*"] \\ \mathbf{Sh}(X_{\text{Zar}}) \arrow[r, "\text{et}"] & \mathbf{Sh}(X_{\text{et}}). \end{tikzcd}$$

Proof. It follows from the commutative diagram:

$$\begin{tikzcd} X_{\text{Zar}} \arrow[d, "f"] & X_{\text{et}} \arrow[l, "\iota"'] \arrow[d, "f"] \\ Y_{\text{Zar}} & Y_{\text{et}}. \arrow[l, "\iota"'] \end{tikzcd}$$

□

2. Frobenii

In this section we will discuss Frobenii, and we begin with the general case where S is a characteristic p scheme, then we specify to case that we need.

2.1. Absolute Frobenius.

First we begin considering the case of an affine scheme

Lemma 2.1. *Let A be a ring of characteristic p, Then the ring map $\rho\colon A \to A$ given by $a \mapsto a^p$ induces the identity(on topological space) on* $\operatorname{Spec}(A) \to \operatorname{Spec}(A)$.

Proof. We need to check that for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$ $\rho^{-1}(\mathfrak{p}) = \mathfrak{p}$.
By set theory we have $\mathfrak{p} \subset \rho^{-1}(\mathfrak{p})$, so to prove the other inclusion we let $x \in \rho^{-1}(\mathfrak{p})$, and that is $x^p \in \mathfrak{p}$, and thus as $\mathfrak{p}$ is a prime ideal, $x \in \mathfrak{p}$, and thus we have proven our claim. □

We will now define the absolute Frobenius.

Definition 2.2. Let X be a scheme of characteristic p. Then, we define the map $F_X : X \to X$ is the identity on topological spaces and $\alpha \mapsto \alpha^q$ on $\mathcal{O}_X \mapsto \mathcal{O}_X$. We call it the *absolute Frobenius.*

Note that by Lemma 2.1 this is well defined, and we note that.

Lemma 2.3. *If $f : X \to Y$ is a map of schemes in characteristic p. Then the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

Proof. As F_X and F_Y are the identity on sets, we only need to check that the sheaf maps commutes. This follows as $\varphi(a)^q = \varphi(a^q)$ for $q = p^f$ in characteristic p, and thus we have proven our claim.

□

We will now prove that this is a particularly well behaved map.

Lemma 2.4. *Let $p > 0$ and let X be a scheme of characteristic p. Then $F_X\colon X \to X$ is a universal homeomorphism.*

Proof. Note that it sufficient to prove the claim for $q = p$. The claim is local on source and target by the usual argument for fiber products. Now we apply [Stacks, Tag 0BRA], and the absolute Frobenius clear satisfies the assumptions of the lemma, thus the claim follows.

□

Later we will prove that the absolute Frobenius induces the identity on étale cohomology, but first we need the notion of relative Frobenius, and some basic fact about it.

2.2. **The relative Frobenius.** Now will move to the definition of the relative Frobenius, which is the one will try to compute later.

Definition 2.5. Let S be a scheme of characteristic p, and let X be a S-scheme then we define $X^{(q)}$ and $F_{X/S}$ from the following diagram.

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, "F_X"] \arrow[ddr] & & \\
& X^{(q)} \arrow[r, "g"] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

$F_{X/S}$ is called the *relative Frobenius*.

To build some intuition about this map which we constructed using the universal property of the pullback, we compute it in the affine case.

Example 2.6. In the case where $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ we obtain that we can translate the above diagram to

$$\begin{tikzcd}
B & & \\
& A \otimes_{F_A,A} B \arrow[ul, dashed, "F_{B/A}"'] & B \arrow[l] \arrow[ull, "F_B"'] \\
& A \arrow[uul] \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

and thus we obtain that $F_{B/A}(a \otimes b) = F_B(a^{1/q}b) = ab^q$.

We will later see that $F_{X/S}$ is a universal homomorphism, and then one can globalize this intuition.

Lemma 2.7. *Let $p >$ and let S be a scheme of characteristic p, and $f\colon X \to Y$ and S-map of schemes. The diagram*

$$\begin{tikzcd}
X \arrow[r, "F_{X/S}"] \arrow[d, "f"'] & X^{(p)} \arrow[d, "f^{(p)}"] \\
Y \arrow[r, "F_{Y/S}"] & Y^{(p)}
\end{tikzcd}$$

commutes.

Proof. By Lemma 2.3 and universal property we have the following commutative diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_{X/S}"'] \arrow[drrrrr, "F_X"] \arrow[ddrr, bend left] \arrow[ddddr] & & & & & \\
& X^{(q)} \arrow[rrrr] \arrow[ddd] \arrow[ddrr, dashed, "f^{(q)}"'] & & & & X \arrow[ddl, "f"] \arrow[ddd] \\
& & Y \arrow[dr, dashed, "F_{Y/S}"] \arrow[drr, "F_Y"] \arrow[ddl] & & & \\
& & & Y^{(q)} \arrow[r] \arrow[dll] & Y \arrow[dr] & \\
& S \arrow[rrrr, "F_S"] & & & & S
\end{tikzcd}$$

and thus the claim follows. ☐

We will now prove that the relative Frobenius also have good behaviour, which justifies our intuition form earlier.

Lemma 2.8. *Let S be a characteristic p scheme, and X and S scheme. $F_{X/S}\colon X \to X^{(q)}$ is a universal homeomorphism.*

Proof. As $F_{X/S}$ is defined by

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, "F_X"] \arrow[ddr] & & \\
& X^{(p)} \arrow[r, "g"] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

by Lemma 2.4 we have that F_S and F_X are universal homeomorphism, and thus so is g. As $F_X = g \circ F_{X/S}$ it follows by [Stacks, Tag 0H2M] is a universal homeomorphism. ☐

Now we will do some preparation for computing the degree.

Lemma 2.9. *Let S be a scheme of characteristic p, and let X be an S scheme such that $X \to S$ is locally of finite type. Then $F_{X/S}$ is finite.*

Proof. As $F_{X/S}$ is a universal homeomorphism it is especially affine [Stacks, Tag 04DE], and thus the claim reduces to the following algebra statement.

Let $A \to B$ be a ring map of finite type of characteristic p, and let $B' = B \otimes_{A,F_A} A$. Then the claim is that $F_{B/A} \colon B' \to B$ given by $b \otimes a \to b^q a$ is finite. Now if $x_1,...,x_n \in B$ generates B over A, then they are integral over B' as $x_i^q = F_{B/A}(x_i \otimes 1)$, and then the claim follows from [Stacks, Tag 02JJ]. □

Lemma 2.10. *Let k be a field of characteristic $p > 0$, X a scheme over k, then X is geometrically reduced if and only if $X^{(q)}$ is reduced.*

Proof. We consider the absolute Frobenius, then $F_k(k) = k^q$, that is it is the same as and embedding of k into $k^{1/q}$. and thus the claim follows from [Stacks, Tag 035X]. □

Proposition 2.11. *Let k be a field of characteristic $p > 0$ and let X be a variety over k, then the following are equivalent.*
1) $X^{(q)}$ is reduced
2) X is geometrically reduced.
3) There exist a $U \subset X$ such that U is smooth over k.

In this case $X^{(q)}$ is a variety and $F_{X/k}$ is a finite dominant map of degree $q^{\deg(X)}$.

Proof. By Lemma 2.10 it follows that 1) and 2) are equivalent. That 2 implies 3 follows from [Stacks, Tag 056V], conversely if we assume 3) then U is geometrically reduced by [Stacks, Tag 038X] and thus so is X by [Stacks, Tag 04KS], and thus we have proven the equivalence of the conditions.
For the second part of the lemma we observe that as $F_{X/k}$ is a homeomorphism, we have that $X^{(q)}$ is a variety, as if one is irreducible, then is the other. We note that $F_{X/k}$ is finite by Lemma 2.9, and is dominant as $F_{X/k}$ is surjective. To compute the degree it suffices to compute the degree of $F_{U/k} \colon U \to U^{(p)}$ as $F_{U/k} = F_{X/k} \mid_U$ by Lemma 2.7.
After shirking U we may assume that there is an étale $h \colon U \to \mathbb{A}_k^n$ by [Stacks, Tag 054L] with $n = \dim(U)$. We note that h is generically finite dominant map of varieties, as étale maps are quasi finite and it is dominant as it is surjective and thus sends generic point to generic point we obtain that it is quasi finite, thus the degree of h is defined. Now by Lemma 2.7 we have the following diagram

$$\begin{tikzcd}
U \arrow[r, "F_{U/k}"] \arrow[d, "h"] & U^{(q)} \arrow[d, "h^{(q)}"] \\
\mathbb{A}^n_k \arrow[r, "F_{\mathbb{A}^n_k/k}"] & (\mathbb{A}^n_k)^{(q)}
\end{tikzcd}$$

Since $h^{(q)}$ is étale by as the base change and then by [Mil80, Corollary I.3.6.], and thus is generically finite dominant map as before.

The degree of $h^{(q)}$ is $[K(U^{(q)}) : K((\mathbb{A}^n_k)^{(q)})]$ which is the same as the degree of $K(U)/K(\mathbb{A}^n_k)$. By multiplicity of degree [Stacks, Tag 02NY] it suffices to compute the degree of $F_{\mathbb{A}^n_k}$. First note that $(\mathbb{A}^n_k)^{(q)} = \mathbb{A}^n_k$ as $(\mathbb{A}^n_k \to \mathrm{Spec}(k)$ is étale, and thus it follows by Lemma 2.15. Secondly note that the relative Frobenius is raising to the q-power in coordinates, and thus is of degree q^n. □

Lemma 2.12. *Let*

$$\begin{tikzcd}
A \arrow[r, "f"] & A \\
B \arrow[u, "i"] \arrow[r, "\mathrm{id}"] & B \arrow[u, "i"']
\end{tikzcd}$$

Be a commutative square in the category of commutative unital rings with f an isomorphism and i the inclusion, then the square is coCartesian.

Proof. We see that the following diagram defined an isomorphism of diagram where the second diagram coCartesian and thus the claim follows.

$$\begin{tikzcd}
 & & A \arrow[r, "\mathrm{id}"] \arrow[dll, "\mathrm{id}"] & A \arrow[dll, "f"] \\
A \arrow[r, "f"] & A & B \arrow[u, "i"] \arrow[r, "\mathrm{id}"] \arrow[dll, "\mathrm{id}"] & B \arrow[u, "i"'] \arrow[dll, "\mathrm{id}"] \\
B \arrow[u, "i"] \arrow[r, "\mathrm{id}"] & B \arrow[u, "i"] & &
\end{tikzcd}$$

□

Proposition 2.13. *Let X_0 be a scheme over $\mathbb{F}_q$, and $X = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, then we have a canonical isomorphism $X^{(q)} \simeq X$ and $F_{X/\overline{\mathbb{F}_q}} = F_{X_0/\mathbb{F}_q} \times \overline{\mathbb{F}_q}$*

Proof. By Lemma 2.12 we obtain that

$$\begin{tikzcd}
\overline{k} \arrow[r, "\alpha\mapsto\alpha^q"] & \overline{k} \\
k \arrow[u, "i"] \arrow[r, "\alpha\mapsto\alpha^q"] & k \arrow[u, "i"']
\end{tikzcd}$$

is coCartesian as the absolute Frobenius on $k = \mathbb{F}_q$ is the identity, and thus we obtain that the following diagram is Cartesian

$$\begin{tikzcd}
\operatorname{Spec}(\overline{k} \otimes_{\overline{k},\sigma} \overline{k}) \arrow[r] \arrow[d] & \operatorname{Spec}(\overline{k}) \arrow[d] \\
\operatorname{Spec}(\overline{k}) \arrow[r, "\sigma"] \arrow[d] & \operatorname{Spec}(\overline{k}) \arrow[d] \\
\operatorname{Spec}(k) \arrow[r, "\sigma"] & \operatorname{Spec}(k)
\end{tikzcd}$$

and thus we obtain that $\operatorname{Spec}(\overline{k} \otimes_{\overline{k},\sigma} \overline{k}) \simeq \operatorname{Spec}(\overline{k} \otimes_{k,\sigma} k)$ and thus we obtain

$$\begin{aligned}
X^{(q)} &= X \times_{\overline{k},\sigma} \overline{k} \\
&\simeq X_0 \times_k \overline{k} \times_{\overline{k},\sigma} \overline{k} \\
&\simeq X_0 \times_k \overline{k} \times_{k,\sigma} k \\
&\simeq X_0 \times_k \overline{k}
\end{aligned}$$

Proving the first claim. The second claim follows is [Görtz, Lemma 26.183(a)].

□

Corollary 2.14. *Let X_0 be a variety over $\mathbb{F}_q$ then the relative Frobenius of $X/\overline{\mathbb{F}_q}$ is given by $F_{X_0} \times \overline{\mathbb{F}_q}$*

Proof. By Proposition 2.13 we know that under base change $\operatorname{Spec}(\overline{\mathbb{F}_q}) \to \operatorname{Spec}(\mathbb{F}_q)$ the relative Frobenius is stable, that it $F_{X/\overline{\mathbb{F}_q}} = F_{X_0/\mathbb{F}_q} \times \overline{\mathbb{F}_q}$ thus is suffuce to prove that $F_{X_0/\mathbb{F}_q} = F_{X_0}$. To do this we consider the following diagram

$$\begin{tikzcd}
X_0 \arrow[dr, "F_{X_0/\mathbb{F}_q}"] \arrow[drr, "F_{X_0}"] \arrow[ddr] & & \\
& X_0 \arrow[r, "\mathrm{id}"] \arrow[d] & X_0 \arrow[d] \\
& \mathrm{Spec}(\mathbb{F}_q) \arrow[r, "\sigma=\mathrm{id}"] & \mathrm{Spec}(\mathbb{F}_q)
\end{tikzcd}$$

and see that the claim follows. □

2.3. **The baffling theorem.**

Lemma 2.15. *For any étale map $f\colon X \to S$ the following diagram is Cartesian.*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S \arrow[r, "F_S"] & S
\end{tikzcd}$$

Proof. Since we have

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, "F_X"] \arrow[ddr, "f"'] & & \\
& X^{(q)} \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

If f is étale then so is f' and then by [Mil80, Corollary I.3.6.] so is $F_{X/S}$, and as Lemma 2.8 $F_{X/S}$ is a universal homeomorphism, and thus by [Stacks, Tag 025G] is an open embedding, but as it is surjective, it is an homeomorphism, and thus proves the claim. □

Now we will see one of the reasons why we do not consider the action by the absolute Frobenius.

Theorem 2.16. *Let X be a scheme of characteristic p, then the absolute Frobenius induces the identity on étale cohomology that is for all $j \geq 0$*

$$F_X^*\colon H^j(X,\Lambda) \to H^j(X,\Lambda)$$

is the identity.

Proof. It follows from Lemma 2.15 that it is the identity as a map of sites, and thus the claim follows. □

2.4. **The arithmetic and geometric Frobenius.** Let $G_k = \mathrm{Gal}(\overline{k}/k)$ for $k = \mathbb{F}_q$, and let X be a scheme over k.

Let $\sigma \in G_k$ since $\mathrm{Spec}(\sigma)\colon \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(\overline{k})$ is a map over k, and thus, we obtain a map $\mathrm{id} \times \mathrm{Spec}(\sigma)\colon X_{\overline{k}} \to X_{\overline{k}}$ and thus by Proposition 1.1 we obtain a map

$$(\mathrm{id} \times \mathrm{Spec}(\sigma))^*\colon H^i(X_{\overline{k}}, \mathbb{Q}_l) \to H^i(X_{\overline{k}}, \mathbb{Q}_l)$$

We note that G_k is topologically generated[Weibel95, Example 6.11.2] by $\sigma_q\colon \alpha \mapsto \alpha^q$. Now we define the geometric and arithmetic Frobenius.

Definition 2.17.
1) The map $\mathrm{id} \times \sigma_q\colon X_{\overline{k}} \to X_{\overline{k}}$ is called the *arithmetic Frobenius.*
2) The map $\mathrm{id} \times \sigma_q^{-1}\colon X_{\overline{k}} \to X_{\overline{k}}$ is called the *geometric Frobenius.*

Now that we have defined the four Frobenii, let us compare them in the case of $\mathbb{A}^1_k$

Example 2.18. $X = \mathrm{Spec}(k[t])$ then we obtain

$$F_{X_{\overline{k}}}\colon \sum a_n t^n \mapsto \sum a_n^p t^{np}$$

$$F_{X/\overline{k}} = F_X \times \mathrm{id}\colon \sum a_n t^n \mapsto \sum a_n t^{np}$$

$$\mathrm{id} \times \sigma_q\colon \sum a_n t^n \mapsto \sum a_n^p t^n$$

$$\mathrm{id} \times \sigma_q^{-1}\colon \sum a_n t^n \mapsto \sum a_n^{1/p} t^n$$

These suggests a certain relationship between these maps, and now we shall make this clear.

Proposition 2.19. $F_{X/\overline{k}} \circ (\mathrm{id} \times \sigma_q) = (\mathrm{id} \times \sigma_q) \circ F_{X/\overline{k}} = F_X \times \sigma_q = F_{X_{\overline{k}}}$

Proof. $F_{X/\overline{k}} \circ (\mathrm{id} \times \sigma_q) = (\mathrm{id} \times \sigma_q) \circ F_{X/\overline{k}} = F_X \times \sigma_q$ so we need to prove that $F_X \times \sigma_q$ is $F_{X_{\overline{k}}}$. The first mentioned is by definition the unique arrow on the diagram

$$\begin{tikzcd}
X_{\overline{k}} \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & \mathrm{Spec}(\overline{\mathbb{F}_q}) \arrow[d, "\sigma_q"] \\
& X_{\overline{k}} \arrow[r] \arrow[d] & \mathrm{Spec}(\overline{\mathbb{F}_q}) \arrow[d] \\
X \arrow[r, "F_X"] & X \arrow[r] & \mathrm{Spec}(\mathbb{F}_q)
\end{tikzcd}$$

But $F_{X_{\overline{k}}}$ also makes this diagram commute, and thus the claim follows. □

Corollary 2.20. *The induced maps on cohomology $F^*_{X/\overline{k}}$ has inverse $(1 \times \sigma_q)^*$.*

Proof. This follows from Proposition 2.19 and Theorem 2.16. □

3. The Frobenius action on $\mathbb{P}^1$

Let us begin with the simplest none trivial case namely that of $\mathbb{P}^1_{\overline{k}}$, where $k = \mathbb{F}_q$.

First Let us consider the action on $H^0(X,\Lambda)$.

Lemma 3.1. *The structure map $\mathbb{P}^n_{\overline{k}} \to \mathrm{Spec}(\overline{k})$ induces an isomorphism $\mathbb{Q}_l \simeq \pi^*\mathbb{Q}_l \simeq \mathbb{Q}_l$*

Proof. We choose a point $\mathrm{Spec}(\overline{k}) \to \mathbb{P}^1_{\overline{k}}$ over $\mathrm{Spec}(\overline{k})$ we obtain a diagram

$$\begin{tikzcd}
\mathrm{Spec}(\overline{k}) \arrow[r, "r"] \arrow[dr, "\mathrm{id}"] & \mathbb{P}^n_{\overline{k}} \arrow[d, "\pi"] \\
& \mathrm{Spec}(\overline{k})
\end{tikzcd}$$

This induces the diagram

$$\begin{tikzcd}
H^0(\mathrm{Spec}(\overline{k}),\mathbb{Q}_l) & H^0(\mathbb{P}^n_{\overline{k}},\mathbb{Q}_l) \arrow[l, "r^*"'] \\
& H^0(\mathrm{Spec}(\overline{k}),\mathbb{Q}_l) \arrow[u, "\pi^*"] \arrow[ul, "\mathrm{id}"]
\end{tikzcd}$$

This yields

$$\begin{tikzcd}
\mathbb{Q}_l & \mathbb{Q}_l \arrow[l, "r^*"'] \\
& \mathbb{Q}_l \arrow[ul, "\mathrm{id}"] \arrow[u, "\pi^*"']
\end{tikzcd}$$

Therefore by linear algebra π^* is an isomorphism, and thus proving the claim. $\square$

Proposition 3.2. *The Frobenius action on $H^0(\mathbb{P}^n_{\overline{k}},\Lambda)$ is the identity.*

Proof. We have the diagram

$$\begin{tikzcd}
\mathbb{P}^n_{\overline{k}} \arrow[rr, "F_{X/\overline{k}}"] \arrow[dr] & & \mathbb{P}^n_{\overline{k}} \arrow[dl] \\
& \mathrm{Spec}(\overline{k}) &
\end{tikzcd}$$

and from this we obtain that

$$\begin{tikzcd}
& H^0(\mathrm{Spec}(k),\mathbb{Q}_l) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
H^0(\mathbb{P}^n_k,\mathbb{Q}_l) \arrow[rr, "F_{X/k}"] & & H^0(\mathbb{P}^n_{\overline{k}},\mathbb{Q}_l)
\end{tikzcd}$$

Which forces that $F^*_{X/\overline{k}}$ is the identity.

$\square$

Proposition 3.3. *The Frobenius action $F^*_{X/\overline{k}}$ on $H^2(\mathbb{P}^1_{\overline{k}},\Lambda)$ is multiplication by q.*

Proof. To accomplish this we begin by computing its action on $H^2(\mathbb{P}^1_{\overline{k}},\mu_m)$. We use the exact sequence $0 \to \mu_l \to \mathbb{G}_m \xrightarrow{(-)^m} \mathbb{G}_m \to 0$, and here we obtain a commutative diagram from the long exact sequence of étale cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbb{P}^1_{\overline{k}},\mathbb{G}_m) \arrow[r] \arrow[d, "F^*_{X/\overline{k}}"] & H^1(\mathbb{P}^1_{\overline{k}},\mathbb{G}_m) \arrow[r] \arrow[d, "F^*_{X/\overline{k}}"] & H^2(\mathbb{P}^1_{\overline{k}},\mu_m) \arrow[r] \arrow[d, "F^*_{X/\overline{k}}"] & 0 \\
0 \arrow[r] & H^1(\mathbb{P}^1_{\overline{k}},\mathbb{G}_m) \arrow[r] & H^1(\mathbb{P}^1_{\overline{k}},\mathbb{G}_m) \arrow[r] & H^2(\mathbb{P}^1_{\overline{k}},\mu_m) \arrow[r] & 0
\end{tikzcd}$$

Which in turn by [Stacks, Tag 03P8] and [Stacks, Tag 03RM] induces

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(\mathbb{P}^1_{\overline{k}}) \arrow[r] \arrow[d, "F^*_{X/\overline{k}}"] & \mathrm{Pic}(\mathbb{P}^1_{\overline{k}}) \arrow[r] \arrow[d, "F^*_{X/\overline{k}}"] & H^2(\mathbb{P}^1_{\overline{k}}, \mu_m) \arrow[r] \arrow[d, "F^*_{X/\overline{k}}"] & 0 \\
0 \arrow[r] & \mathrm{Pic}(\mathbb{P}^1_{\overline{k}}) \arrow[r] & \mathrm{Pic}(\mathbb{P}^1_{\overline{k}}) \arrow[r] & H^2(\mathbb{P}^1_{\overline{k}}, \mu_m) \arrow[r] & 0
\end{tikzcd}$$

and if we prick a generator of $\mathrm{Pic}(\mathbb{P}^1_{\overline{k}})$ it image in $H^2(\mathbb{P}^1_{\overline{k}}, \mu_m)$ will be a generator and thus it suffices to compute the action on $\mathrm{Pic}(\mathbb{P}^1_{\overline{k}})$, which by [Har-AG, Proposition II.6.9] and Proposition 1.3 is the degree of $F^*_{X/\overline{k}}$, which by Proposition 2.11 is q. Now as $\overline{k}$ is algebraically closed we have an isomorphism of Λ and μ_m, and thus we obtain the following commutative diagram

$$\begin{tikzcd}
H^2(\mathbb{P}^1_{\overline{k}}, \Lambda) \arrow[r, "\sim"] \arrow[d, "F^*_{X/k}"] & H^2(\mathbb{P}^1_{\overline{k}}, \mu_m) \arrow[d, "F^*_{X/k}"] \\
H^2(\mathbb{P}^1_{\overline{k}}, \Lambda) \arrow[r, "\sim"] & H^2(\mathbb{P}^1_{\overline{k}}, \mu_m)
\end{tikzcd}$$

and thus by the cohomology of curves [Stacks, Tag 03RQ] this is

$$\begin{tikzcd}
\mathbb{Z}/m\mathbb{Z} \arrow[r, "\sim"] \arrow[d, "F^*_{X/\overline{k}}"] & \mathbb{Z}/m\mathbb{Z} \arrow[d, "\cdot q"] \\
\mathbb{Z}/m\mathbb{Z} \arrow[r, "\sim"] & \mathbb{Z}/m\mathbb{Z}
\end{tikzcd}$$

and thus the action is $\cdot q$. □

We shall see later that this count points on $\mathbb{P}^1_k$.

4. Twist and fundamental theorems

4.1. The definition. Let us begin with some motivation for the definition of the Tate twist. We see that σ_q acts on $\mathbb{Z}/l^n\mathbb{Z}$ by $a \mapsto a^q$ and thus acts on $\mathbb{Q}_l(1) := \lim_n \mu_{l^n}$ by

$$\sigma_q((\mu_m)_{m\geq 1}) = (\mu_m^q)_{n\geq 1} = q(\mu_m)_{m\geq 1}$$

where the last is when choosing a basis. We know define

Definition 4.1. Let $\mathcal{F}$ be an Λ-module then we define

$$\mathcal{F}(n) = \mathcal{F} \otimes \Lambda(n)$$

where

$$\Lambda(n) = \begin{cases} \mu_m^{\otimes n} & \text{if } n \geq 0 \\ \mathrm{Hom}_\Lambda(\mu_m^{\otimes -n}, \Lambda) & \text{if } n < 0 \end{cases}$$

Remark 4.2. When taking limits of twists here recovers the twist on $\mathbb{Q}_l$.

4.2. **Some statements of étale cohomology.** Now we state a few central theorems of étale cohomology, and we emphasize their equivariance.

Theorem 4.3. *[Poincaré duality] Let X be a smooth variety over a separably closed field k of dimension d, and $\mathcal{F} \in \mathbf{Sh}^{a,con}(X_{\mathrm{et}})$ then the cup product paring*

$$H_c^r(X,\mathcal{F}) \times H^r(X,\check{\mathcal{F}}(d)) \to H_c^{2d}(X,\lambda(d)) \simeq \Lambda$$

is a perfect G_k-equivariant paring.

Proof. [SGA4, Théorème 18.3.2.5] □

Proposition 4.4. *[Gysin sequence] Let $Z \subset X$ be a closed subscheme over a field k, where Z and X are smooth and Z has pure codimension c in X and let $U = X \setminus Z$ then for $0 \leq r \leq 2c-1$ and for $\mathcal{F}$ a locally constant constructible sheaf on X_{et} the restriction map induces an isomorphism*

$$H^r(X_{et},\mathcal{F}) \simeq H^r(U_{et},\mathcal{F}\mid_U)$$

and a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^{2c-1}(X_{et},\mathcal{F}) \arrow[r] & H^{2c-1}(U_{et},\mathcal{F}\mid_U) \arrow[r] & H^0(Z,\mathcal{F}(-c)) \arrow[dll] \\ & H^{2c}(X_{et},\mathcal{F}) \arrow[r] & \ldots & \end{tikzcd}$$

with the connecting map being G_k-equivariant.

Proof. This follows from Purity [SGA4, Théorème 16.3.7.] □

Theorem 4.5. *[Lefschetz fixed-point formula] Let X be a variety over a finite field k of dimension d, and let $F_{X/\overline{k}}$ be the relative Frobenius, Then*

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2d} (-1)Tr(F_{X/k}^* \mid H^i(X_{\overline{k}},\mathbb{Q}_l)$$

There $\#X(\mathbb{F}_q)$ denotes the rational points of the variety X.

Proof. [SGA4.5, Théorème 3.2] □

5. Computing the Cohomology

We wish to compute the cohomology of projective space, but to do this we do some preparatory work, namely computing the cohomology of affine space

Proposition 5.1. *Let $\overline{k}$ be algebraically closed.*

$$H^i(\mathbb{A}^1_{\overline{k}},\mu_m) \simeq \begin{cases} \mu_m & i = 0 \\ 0 & i > 0 \end{cases}$$

Proof. First note that by [Stacks, Tag 03RM] $H^i(\mathbb{A}^1_{\overline{k}},\mathbb{G}_m) = 0$ for $i \geq 2$ and as $H^1(\mathbb{A}^1_{\overline{k}},\mathbb{G}_m) \simeq Pic(\mathbb{A}^1_{\overline{k}}) \simeq 0$. We use the short exact sequence

$$0 \to \mu_m \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

to obtain a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathbb{A}^1_{\overline{k}},\mu_m) \arrow[r] & H^0(\mathbb{A}^1_{\overline{k}},\mathbb{G}_m) \arrow[r, "(-)^m"] & H^0(\mathbb{A}^1_{\overline{k}},\mathbb{G}_m) \arrow[dll] \\ & H^1(\mathbb{A}^1_{\overline{k}},\mu_m) \arrow[r] & 0 & \\ & & & \cdots \end{tikzcd}$$

As $H^0(\mathbb{A}^1_{\overline{k}},\mathbb{G}_m) = \mathcal{O}_{\mathbb{A}^1_{\overline{k}}}(\mathbb{A}^1_{\overline{k}})^* = \overline{k}^*$ we obtain that

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathbb{A}_{\overline{k}},\mu_m) \arrow[r] & \overline{k}^* \arrow[r, "(-)^m"] & \overline{k}^* \arrow[dll] \\ & H^1(\mathbb{A}_{\overline{k}},\mu_m) \arrow[r] & 0 & \\ & & & \cdots \end{tikzcd}$$

Now as $\overline{k}k$ is algebraically closed we know that m-roots exist, that is $(-)^m \colon \overline{k}^* \to \overline{k}^*$ is surjective and thus the long exact sequence factors,

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{A}_{\overline{k}},\mu_m) \arrow[r] & \overline{k}^* \arrow[r, "(-)^m"] & \overline{k}^* \arrow[dl] \\
 & & 0 \arrow[dll] & \\
H^1(\mathbb{A}_{\overline{k}},\mu_m) \arrow[r] & 0 & & \\
 & & & \cdots
\end{tikzcd}$$

And thus $H^1(\mathbb{A}^1_{\overline{k}},\mu_m) \simeq 0$ and $H^0(\mathbb{A}^1_{\overline{k}},\mu_m) = \ker((-)^m \colon \overline{k}^* \to \overline{k}^*) = \mu_m)$. We then proving the claim. □

Corollary 5.2.

$$H^i(\mathbb{A}^k_{\overline{k}},\mu_m) \simeq \begin{cases} \mu_m & i = 0 \\ 0 & i > 0 \end{cases}$$

Proof. Combine previous with [SGA4, Théorème 5.4.3.] □

And thus we obtain

Corollary 5.3.

$$H^i(\mathbb{A}^k_{\overline{k}},\Lambda) \simeq \begin{cases} \mathbb{Z}/m\mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

Proof. Corollary 5.2 and none canonical isomorphism, □

Now we may commute the cohomology of projective space.

Theorem 5.4. *Let $\overline{k}$ be a algebraically closed, then*

$$H^i(\mathbb{P}^n_{\overline{k}},\Lambda) \simeq \begin{cases} \mathbb{Z}/m\mathbb{Z}(-\frac{i}{2}) & i \textit{ is even and } i \leq 2n \\ 0 & \textit{else} \end{cases}$$

Proof. We use the Gysin sequence for the pair $(\mathbb{P}^n_{\overline{k}},\mathbb{P}^{n-1}_{\overline{k}})$ for $c = 1$ Proposition 5.1 and by induction. Here we obtain that

$$H^0((\mathbb{P}^n_{\overline{k}},\Lambda) \simeq H^0(\mathbb{A}^k_{\overline{k}},\Lambda) \simeq \mathbb{Z}/m\mathbb{Z}$$

Now we Gysin sequence is

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbb{P}^n_{\overline{k}},\Lambda) \arrow[r] & H^1(\mathbb{A}^n_{\overline{k}},\Lambda) \arrow[r] & \ldots. \arrow[dlll] \\
H^i(\mathbb{P}^n_{\overline{k}},\Lambda \arrow[r] & H^i(\mathbb{A}^n_{\overline{k}},\Lambda \arrow[r] & H^{i-2}(\mathbb{P}^{n-1}_{\overline{k}},\Lambda(-1)) \arrow[r] & \ldots
\end{tikzcd}$$

As $H^i(\mathbb{A}^k_{\overline{k}},\Lambda)$ vanish for $i > 0$ we obtain that $H^{2i}(\mathbb{P}^n_{\overline{k}},\Lambda) \simeq H^{2i-2}(\mathbb{P}^{n-1}_{\overline{k}},\Lambda(-1))$ now by the projection formula [Mil13, Proposition 22.5] we obtain That this is isomorphic to $H^{2i-2}(\mathbb{P}^{n-1}_{\overline{k}},\Lambda)(-1)$, and thus we combining we obtain

$$\begin{aligned}
H^i(\mathbb{P}^n_{\overline{k}},\Lambda) &\simeq H^{i-2}(\mathbb{P}^{n-1}_{\overline{k}},\Lambda)(-1) \\
&\simeq \begin{cases} \mathbb{Z}/m\mathbb{Z}(-\frac{i-2}{2})(-1) & i \text{ is even and } i \leq 2n \\ 0(-1) & \text{else} \end{cases} \\
&\simeq \begin{cases} \mathbb{Z}/m\mathbb{Z}(-\frac{i}{2}) & i \text{ is even and } i \leq 2n \\ 0 & \text{else} \end{cases}
\end{aligned}$$

□

Theorem 5.5. *If we consider $k = \mathbb{F}_q$, then Frobenius action on $H^{2i}(\mathbb{P}^n_{\overline{k}},\Lambda)$ is multiplication by q^i for $0 \leq i \leq n$, and trivial otherwise.*

Proof. We pressed by induction and for the base case $n = 0$ we only have the zeroth cohomology group and thus it follows by Proposition 3.2 From the proof Theorem 5.4 we have the identification $H^{2i}(\mathbb{P}^n_{\overline{k}},\Lambda) \simeq H^{2i-2}(\mathbb{P}^{n-1}_{\overline{k}},\Lambda)(-1)$ and since the isomorphisms here are Galois equivariant, our claim follows. □

Finally we can arrive at the computation that we wanted.

Corollary 5.6. *Let $k = \overline{\mathbb{F}_q}$, then*

$$H^i(\mathbb{P}^n_{\overline{k}},\mathbb{Q}_l) \simeq \begin{cases} \mathbb{Q}_l(-\frac{i}{2}) & i \textit{ is even and } i \leq 2n \\ 0 & \textit{else} \end{cases}$$

and the Frobenius action on $H^{2i}(\mathbb{P}^n_{\overline{k}},\mathbb{Q}_l)$ is multiplication by q^i for $0 \leq i \leq n$, and trivial otherwise.

Proof. Follows by taking limits in the of in Theorem 5.4 and Theorem 5.5. □

Using the preceding corollary and the Lefschetz fix point formula we obtain the following.

Corollary 5.7. *The number of points on* $\mathbb{P}^n_{\mathbb{F}_q}$ *is* $\dfrac{q^{n+1}-1}{q-1}$.

Proof. This follows by Corollary 5.6 and Theorem 4.5 □

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