SIX FUNCTOR FORMALISM OF ANIMA AND THE NORM MAP

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Contents

Introduction

In these notes we will define the norm map for a general six functor formalism, and after this we will restrict ourselves to the case of anima, where we will compute the norm map, and here we will obtain a universal property, that will help us prove that the norm map is an equivalence for compact anima.

Furthermore we will prove the Künneth formula and Mayer-Vietoris.

1. Norm map in general six functor formalism

In this subsection we assume that $\mathcal{C}$ is an ∞-category which admits finite limits and that is a six functor formalism $D\colon \mathbf{Corr}(\mathcal{C}) \to \mathbf{LPr}_{st}$, in the sense of [Mann22, Definition A.5.7.]. We first define the so–called Klein—Spivak dualizing object.

Construction 1.1. Let $f\colon Y \to X$ be a map in the ∞-category $\mathcal{C}$.

We consider the diagram

$$\begin{tikzcd}
Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_Y"] \arrow[ddr, bend right, "\mathrm{id}_Y"'] & & \\
& Y \times_X Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"] & X.
\end{tikzcd}$$

and define the *Klein Spivak dualizing object* to be $D_f := p_{1*}\Delta_!(1_X) \in D(Y)$, where 1_Y is the tensor unit in $D(Y)$.

Now having this object, we define the norm map.

Construction 1.2. Let $f\colon Y \to X$ be a map in the ∞-category $\mathcal{C}$. We wish to construct a natural transformation

$$f_!(- \otimes D_f) \to f_*(-).$$

As in Construction 1.1 we have the diagram

$$\begin{tikzcd}
Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_Y"] \arrow[ddr, bend right, "\mathrm{id}_Y"'] & & \\
& Y \times_X Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"] & X.
\end{tikzcd}$$

as f_* is right adjoint to f^* we have an equivalence of mapping anima

$$\mathrm{Map}_{D(X)}(f_!(\mathcal{F} \otimes D_f), f_*(\mathcal{F})) \simeq \mathrm{Map}_{D(Y)}(f^* f_!(\mathcal{F} \otimes D_f), \mathcal{F})$$

for $\mathcal{F} \in D(Y)$.

Thus it suffices to give a map $f^*f_!(-\otimes D_f) \to (-)$, and we obtain

$$
\begin{aligned}
f^*f_!(-\otimes p_{1*}\Delta_!(1_X) &\simeq p_{2!}(p_1^*(-\otimes p_{1*}\Delta_!(1_X))) && (1.3)\\
&\simeq p_{2!}(p_1^*(-)\otimes p_1^*p_{1*}\Delta_!(1_X))) && (1.4)\\
&\xrightarrow{\varepsilon} p_{2!}(p_1^*(-)\otimes \Delta_!(1_X))) && (1.5)\\
&\simeq (p_{2!}\Delta_!)(\Delta^*p_1^*\otimes 1_X) && (1.6)\\
&\simeq (-) && (1.7)
\end{aligned}
$$

where we used base change in Equation (1.3), that p_1^* is symmetric monoidal in Equation (1.4), the counit for $p_1^* \dashv p_{1*}$ in Equation (1.5) and the projection formula in Equation (1.6). We call this map the *adjoint norm map* and we denote it by $\widetilde{\mathrm{Nm}}_f$. Furthermore we call the mate of $\widetilde{\mathrm{Nm}}_f$ under the adjunction $f^* \dashv f_*$ the *norm map*, and we denote it by Nm_f.

2. Norm map in the six functor formalism for anima

In this subsection we will characterise the norm map in the case of anima and for the map $p\colon X \to 1$, where X is an anima and 1 is the final anima. Recall that in the six functor formalism for anima we have that $D(X) = \mathbf{Fun}(X, \mathrm{Sp})$. All of the following proofs are due to [Cno23].

We begin with a few preliminary remarks that we need for identifying our norm map with a map that has a certain universal property.

Definition 2.1. Let X, Y be anima, and $p_X : X \to 1$ and $p_Y : Y \to 1$ the unique maps to the final anima. In the following diagram

$$
\begin{tikzcd}
X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\
X \arrow[r, "p_X"] & 1
\end{tikzcd}
$$

We define $\mathcal{F} \boxtimes \mathcal{G} := p_1^*(\mathcal{F}) \otimes p_2^*(\mathcal{G})$, and call it the *external tensor product* of $\mathcal{F} \in D(X)$ and $\mathcal{G} \in D(Y)$.

Remark 2.2. There is a canonical diagram

$$\begin{tikzcd} D(X) \otimes D(X) \arrow[r, "\boxtimes"] \arrow[dr, "\otimes"] & D(X \times X) \arrow[d, "\Delta^*"] \\ & D(X). \end{tikzcd}$$

Lemma 2.3. *If X is an anima and $\mathcal{F} \in \mathbf{Fun}(X, \mathrm{Sp})$, then we have*

$$(p^*)^X \simeq p_1^*$$

$$(p_!)^X \simeq p_{2!}$$

$$(- \otimes \mathcal{F})^X \simeq - \otimes p_1^* \mathcal{F}$$

Proof. First we note that upper star is precomposition, thus adding a free variable is the same as adding it in p, and thus we obtain $(p^*)^X \simeq p_1^*$, and then by adjointness as $\mathbf{Fun}(X,-)$ preserves adjoints, we obtain that $(p_!)^X \simeq p_{1!}$

For the last one we note that we can write tensoring with $\mathcal{F}$ as

$$D(X) \xrightarrow{(\mathrm{id}\,,\mathcal{F})} D(X) \otimes D(X) \xrightarrow{\boxtimes} D(X \times X) \xrightarrow{\Delta^*} D(X)$$

Note that when taking $(-)^X$ on the first map we obtain $(\mathrm{id}\,, \mathrm{pr}_2^* \mathcal{F})$ and thus the claim follows.

∎

Corollary 2.4. *Given $X \in An$ evaluation at $\Delta_! 1_X$*[1] *induces an equivalence*

$$\mathrm{Map}(p^* p_!(- \otimes D_p), \mathrm{id}_{\mathrm{Sp}^X}) \simeq \mathrm{Map}_{\mathrm{Sp}^{X \times X}}(p_1^* D_p, \Delta_! 1_X)$$

of mapping anima.

Proof. By Lemma 2.3 evaluating $p^* p_!(- \otimes D_p)$ at $\Delta_! 1_X$ gives

$$p_1^* p_{1!}(\Delta_! 1_X \otimes p_2^* D_p)) \overset{\text{projection formula}}{\simeq} p_1^* p_{1!}(\Delta_!(\Delta^*(p_2^* D_p)) \simeq p_1^* D_p,$$

then by Corollary A.5 we obtain our claim. ∎

Lemma 2.5. *The adjoint norm map $\widetilde{\mathrm{Nm}}_p \colon p^* p_!(- \otimes D_X) \to \mathrm{id}_{\mathrm{Sp}^X}$ is equivalent to the map that corresponds to the map $p_1^* p_{1*} \Delta_! 1_X \to \Delta_! 1_X$, which is induced by the counit map for the adjunction $p_1^* \dashv p_{1*}$ under the equivalence in Corollary 2.4.*

[1] As defined in Definition A.3

Proof. The map $p_1^*p_{1*}\Delta_!1_X \to \Delta_!1_X$ induced by the counit map for the adjunction $p_1^* \dashv p_{1*}$ is by Proposition A.4 sent to

$$p_{2!}(p_1^*(-) \otimes p_1^* D_p) \to p_{2!}(p_1^*(-) \otimes \Delta_! 1_X).$$

Which is exactly the induced map of Construction 1.2 thus proving our claim. $\square$

Now we can give the universal property of $\widetilde{\mathrm{Nm}}_p$, which we shall use to obtain the universal property of Nm_p.

Theorem 2.6. *For a colimit preserving functor $F\colon \mathrm{Sp}^X \to \mathrm{Sp}$ the composition*

$$\alpha\colon \operatorname{Map}(F,p_!(-\otimes D_p) \xrightarrow{p^*\circ} \operatorname{Map}(p^*F,p^*p_!(-\otimes D_p)) \xrightarrow{\widetilde{\mathrm{Nm}}_p} \operatorname{Map}(p^*F,\mathrm{id}_{\mathrm{Sp}^X})$$

is an equivalence of anima.

Proof. We we wish to show that we have a diagram

$$\begin{tikzcd}
\operatorname{Map}(F,p_!(-\otimes D_p)) \arrow[r, "p^*\circ"] \arrow[d, "ev"] & \operatorname{Map}(p^*F,p^*p_!(-\otimes D_p)) \arrow[r, "\widetilde{\mathrm{Nm}}_p\circ"] \arrow[d, "ev"] & \operatorname{Map}(p^*F,\mathrm{id}_{\mathrm{Sp}^X}) \arrow[d, "ev"] \\
\operatorname{Map}(F^X(\Delta_!1_X),D_p) \arrow[r, "p_1^*"] & \operatorname{Map}(p_1^*F^X(\Delta_!1_X),p_1^*D_p) \arrow[r, "\varepsilon_p^*\circ"] & \operatorname{Map}(p_1^*F_X(\Delta_!1_X),\Delta_!1_X))
\end{tikzcd}$$

where ε_{p_1} is the counit for $p_1^* \dashv p_{1*}$ adjunction.

The the right square is by naturality. We have to check that the bottom map is indeed p_1^*. By Proposition A.4 we have an equivalence $\mathrm{Sp}^Y \simeq \mathbf{Fun}^L(\mathrm{Sp}^Y,\mathrm{Sp})$ given by β sending $\mathcal{F} \mapsto p_{Y!}(\mathcal{F}\otimes -)$, where $Y \in \mathbf{An}$. We wish to check that we have the following diagram, for $f\colon Y \to Z$ in $\mathbf{An}$.

$$\begin{tikzcd}
\mathrm{Sp}^Y \arrow[r, "\beta"] & \mathbf{Fun}^L(\mathrm{Sp}^Y,\mathrm{Sp}) \\
\mathrm{Sp}^Z \arrow[r, "\beta"] \arrow[u, "f^*"] & \mathbf{Fun}^L(\mathrm{Sp}^Z,\mathrm{Sp}) \arrow[u, "(f_!)^*"']
\end{tikzcd}$$

If we apply the map to f^* then we obtain

$$p_{Y!}(f^*\mathcal{G}\otimes -)$$

and for the other one we get $(f_!)^*(p_{Z!}(\mathcal{G}\otimes -) \simeq p_{Z!}(\mathcal{G}\otimes f_!(-)) \simeq p_{Z!}f_!(f^*\mathcal{G}\otimes -) \simeq p_{Y!}(f^*\mathcal{G}\otimes -)$ proving that we have the diagram. By applying it to $f = p_1$ and $Z = X\times X$ and $Y = X$, we obtain the following diagram

$$\begin{tikzcd}
\mathrm{Sp}^X \arrow[r, "\beta"] & \mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp}) \\
\mathrm{Sp}^{X\times X} \arrow[u, "p_1^*"] \arrow[r, "\beta"] & \mathbf{Fun}^L(\mathrm{Sp}^{X\times X}, \mathrm{Sp}) \arrow[u, "(p_{1!})^*"']
\end{tikzcd}$$

Thus if we can prove that the following diagram commutes

$$\begin{tikzcd}
\mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp}) \arrow[r, "p^*\circ"] \arrow[dr, "(p_{1!})^*"] & \mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp}^X) \arrow[d, "\beta\circ"] \\
& \mathbf{Fun}^L(\mathrm{Sp}^X, \mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp}))
\end{tikzcd}$$

Then by combining the two diagrams we obtain the following diagram

$$\begin{tikzcd}
& \mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp}) \arrow[r, "p^*\circ"] \arrow[dr, "(p_{1!})^*"] & \mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp}^X) \arrow[d, "\beta"] \\
\mathrm{Sp}^X \arrow[ur] \arrow[dr, "p_1^*"] & & \mathbf{Fun}^L(\mathrm{Sp}^X, \mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp})) \\
& \mathrm{Sp}^{X\times X} \arrow[ur] &
\end{tikzcd}$$

reduced to proving that the composition in the top of the diagram is precomposition with $p_{1!}$.

We check this pointwise and here we obtain that $F \mapsto p^*\circ F \mapsto p_!(p^*\circ F(-)\otimes(-))$(note the two variables here are different). We prove that this is equivalent to $F(-)\otimes p_!(-)$.

We define a map $p_!(F(-)\otimes(-)) \to F(-)\otimes p_!(-)$ using the adjunction $p_! \dashv p^*$ to be the mate of $p^*F(-)\otimes(-) \to p^*F(-)\otimes p^*p_!(-)$ which is the identity in one first component and the second component it is the unit. Since both sides preserves colimits, when evaluating F we obtain a spectra and thus is suffices to check the isomorphism on $\mathbb{S}$ and there it becomes

$$p^*\mathbb{S}\otimes\mathbb{S} \to p^*\mathbb{S}\otimes\mathbb{S}$$

which corresponds to the identity and thus we have our equivalence.

On the other side we by construction obtain

$$(F \circ p_{1!})(p_1^*(-) \otimes p_2^*(-)) \simeq F((-) \otimes p_{1!}p_2^*(-)) \tag{2.7}$$

$$\simeq F((-) \otimes p^* p_!(-) \tag{2.8}$$

$$\simeq F(-) \otimes p_!(-) \tag{2.9}$$

where the last equivalence follows from $F(- \otimes p^*(-)) \simeq F(-) \otimes (-)$, which follows from checking on the sphere spectrum, and we may check one points because of the equivalence $\mathbf{Fun}^L(\mathrm{Sp}, \mathcal{D}) \simeq \mathcal{D}$, and thus we have desired the diagram.

The left square commutes by functoriality. The composition at bottom of the diagram is an equivalence as ε_{p_1} is the counit [Land21, Proposition 5.1.10], and the claim follows. □

Now we can finally obtain the universal property of Nm_p.

Corollary 2.10. *The norm map* $\mathrm{Nm}_p \colon p_!(- \otimes D_p) \to p_*(-)$ *is final in the* ∞*-category* $\mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp})_{/p_*}$.

Proof. By construction we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(F, p_!(- \otimes D_p) \arrow[d] \arrow[dr, "\alpha"] & \\
\mathrm{Map}(F, p_*(-)) \arrow[r] & \mathrm{Map}(p^*F, (-))
\end{tikzcd}$$

The right map is an equivalence by Theorem 2.6, and the bottom map is an equivalence as it is the adjunction map, and thus by two out of three the last map is an equivalence and thus we have proven our claim.

□

What this universal property is saying is that the norm map approximates the best colimit preserving functor of $f_*(-)$ by $f_!(- \otimes D_p)$. One can think of this as how close are we from having Poincaré duality in this more general setting.

Remark 2.11. This indeed suffices to identify with the dualizing obejct of [NikSch18]. In their proof of [NikSch18, Theorem 4.I] they prove that any colimit preserving functor is of the form $\mathrm{colim}_X(- \otimes D)$, for some unique D. They identify with their dualizing object, for the universal property that we just proved for our dualizing obejct in Corollary 2.10, and thus they must be equivalent.

3. The norm map in the case of compact anima

Now we wish to use what we have developed to prove that when X is a compact anima, the norm map is an equivalence, and then we will obtain some generalised form of Poincaré duality. The strategy to doing so will be to prove that p_* preserves small colimits when X is a compact anima and then use the universal property that we described in the previous section.

We begin by proving that when X is a finite anima, then p_* preserves small colimits.

Lemma 3.1. *If X is a finite anima, then the functor $p_*\colon D(X) \to D(1) \simeq \mathrm{Sp}$ preserves small colimits.*

Proof. We wish to prove that p_* preserves colimits, but we are in a stable infinity category, so to preserve with finite limit is equivalent to preserving with finite colimits by [Lur17, Proposition 1.1.4.1] and as colimits preserves colimits and p_* is a finite limit, thus we have proven our claim. □

Now we can prove that p_* preserves colimits when X is a compact anima, but first we shall state a result that we will need to prove this.

Proposition 3.2. *Let $\mathcal{C} \in \widehat{\mathbf{Cat}}_\infty^L$ The assignment $(f^*\colon \mathcal{C} \to \mathcal{D}) \mapsto (f_*f^*\colon \mathcal{C} \to \mathcal{C})$ promotes a functor $(\widehat{\mathbf{Cat}}_\infty^L)_{\mathcal{C}/} \to \mathbf{Fun}(\mathcal{C},\mathcal{C})$.*

Proof. This is Corollary B.4. □

Lemma 3.3. *Let X be a compact anima and $p\colon X \to 1$, then p_* preserves small colimits.*

The following proof is due to Li He.

Proof. As X is compact it is by [Lur09, Proposition 5.3.4.17] a retract of a finite anima Y and we obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[dr] & Y \arrow[r, "r"] \arrow[d] & X \arrow[dl] \\
& 1 &
\end{tikzcd}$$

with $r \circ s \simeq \mathrm{id}$.

We want a composition of natural transformations such that $\mathrm{id} \to r_* r^* \to r_* s_* s^* r^* \simeq \mathrm{id}$ is the identity. We have the diagram.

$$\begin{tikzcd} & D(X) \arrow[dl, "\mathrm{id}"] \arrow[d, "r^*"] \arrow[dr, "(rs)^* \simeq \mathrm{id}"] & \\ D(X) & D(Y) \arrow[l, "s^*"] & D(X) \arrow[l, "r^*"] \arrow[ll, bend left, "\mathrm{id}"] \end{tikzcd}$$

Now by the functoriality of Proposition 3.2 it is sent to the diagram

$$\begin{tikzcd} \mathrm{id} \arrow[r] \arrow[dr, "\mathrm{id}"] & r_* r^* \arrow[d] \\ & r_* s_* s^* r^* \simeq \mathrm{id} \end{tikzcd}$$

thus we have that the second map is a retraction.

We will now prove that it preserves small colimits. We obtain the following commutative diagram

$$\begin{tikzcd} u_i \arrow[r] \arrow[d] & r_* r^* u_i \arrow[r] \arrow[d] & r_* s_* s^* r^* u_i \arrow[d] \\ \mathrm{colim}_i\, u_i \arrow[r] & r_* r^* \,\mathrm{colim}_i\, u_i \arrow[r] & r_* s_* s^* r^* \,\mathrm{colim}_i\, u_i. \end{tikzcd}$$

If we apply p_* to the entire diagram, and take colimit on the first row we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{colim}_i\, p_* u_i \arrow[r] \arrow[d] & \mathrm{colim}_i\, p_* r_* r^* u_i \arrow[r] \arrow[d] & \mathrm{colim}_i\, p_* r_* s_* s^* r^* u_i \arrow[d] \\ p_* \,\mathrm{colim}_i\, u_i \arrow[r] & p_* r_* r^* \,\mathrm{colim}_i\, u_i \arrow[r] & p_* r_* s_* s^* r^* \,\mathrm{colim}_i\, u_i \end{tikzcd}$$

as $p_* r_* \simeq q_*$ and $rs \simeq id$ we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{colim}_i\, p_* u_i \arrow[r] \arrow[d] & \mathrm{colim}_i\, q_* r^* u_i \arrow[r] \arrow[d] & \mathrm{colim}_i\, p_* u_i \arrow[d] \\ p_* \,\mathrm{colim}_i\, u_i \arrow[r] & q_* r^* \,\mathrm{colim}_i\, u_i \arrow[r] & p_* \,\mathrm{colim}_i\, u_i. \end{tikzcd}$$

Since Y is a finite anima the middle map is an equivalence by Lemma 3.1. It follows as equivalences are preserved by retractions [Lur09, Remark 5.5.4.6.] that the right map is an equivalence, thus proving our claim. □

And now finally we can prove that the norm map is an equivalence when X is a compact anima.

Proposition 3.4. *Let X be a compact anima and let $p\colon X \to 1$, then the norm map $\mathrm{Nm}_p\colon p_!(- \otimes D_p) \to p_*(-)$ is an equivalence, thus proving our claim.*

Proof. Since p_* preserves colimits by Lemma 3.3, as it exhibits the source as the universal colimit preserving approximation Corollary 2.10, we obtain that it is indeed an equivalence, and thus proving the claim. □

Second proof of Proposition 3.4. Consider $1 \xrightarrow{1_X} X \in D(X)$ as an element of $\mathrm{Hom}_{\mathbf{LZ}_D}(1,X)$. We wish to prove that it has a right adjoint As X is compact p_* preserves colimits by Lemma 3.3 and as the Fourier—Mukai transform is an equivalence by Proposition A.4, we obtain an equivalence $p_* \simeq p_!(- \otimes K)$ for some $K \in D(X)$, this implies that $X \xrightarrow{K} 1$ is the right adjoint of 1_X, thus we have proven that 1_X is p-proper, and thus by Proposition C.5 the norm map is an equivalence. □

Proposition 3.5. *Let $f\colon Y \to X$ be a map in* **An** *then the norm map $\mathrm{Nm}_f\colon f_!(- \otimes D_f) \to f_*(-)$ is an equivalence if and only if the norm map at all fibers of f is an equivalence.*

Proof. First if we assume that Nm_f is an equivalence, and if we are given a commutative diagram

$$\begin{tikzcd} Y_x \arrow[r, "s'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ x \arrow[r, "s"] & X \end{tikzcd}$$

as the norm map by assumption is an equivalence it follows by Proposition C.5 that 1_X is f-proper and as f-proper is stable under base change [Sch22, Remark 6.2] that the norm map for $\mathrm{Nm}_{f'}$ is an equivalence by [Sch22, Proposition 6.9].

Conversely assume that all fibers the norm map is an equivalence. First note that by the Yoneda lemma [Lur09, Proposition 5.1.3.1] it suffices to prove that

$$\mathrm{Map}(C,f_!(B \otimes D_f)) \to \mathrm{Map}(C,f_*(B))$$

the induced map is an equivalence for all $C \in D(X)$.

By [HopLur13, Lemma 4.3.8] we may reduce to object of the form $s_! A$ for $A \in \mathrm{Sp}$, and thus we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(s_!A, f_!(B \otimes D_f)) \arrow[r] \arrow[d] & \mathrm{Map}(s_!A, f_!(B)) \arrow[d] \\
\mathrm{Map}(A, s^* f_!(B \otimes D_f)) \arrow[d] & \mathrm{Map}(A, s^* f_!(B)) \arrow[d] \\
\mathrm{Map}(A, f'_! s'^*(B \otimes D_f) = \mathrm{Map}(A, f'_!(s'^*(B) \otimes D_{f'}) \arrow[r] & \mathrm{Map}(A, f'_* s'^*(B))
\end{tikzcd}$$

Where by assumption and base change all maps but the top one is an equivalence, but then the last is also an equivalence by 3 out of 4, and thus we have proven our claim. $\square$

Theorem 3.6. *Let $f: Y \to X$ be a map in* **An** *with compact fibers, then the norm map is an equivalence.*

Proof. This follows directly by combining Proposition 3.4 and Proposition 3.5, and thus we have the claim. $\square$

This theorem we may interpret as a generalised form of twisted Poincaré duality, as p_* is cohomology and $p_!$ is homology, and when we are given an equivalence $1[-n] \simeq D_p$ we may obtain a more classical Poincaré duality statement, as we recall that p_* is cohomology and $p_!$ is homology.

Corollary 3.7. *Let $f: Y \to X$ be a map in* **An** *with compact fibers, then an equivalence $1[-n] \simeq D_p$ determines an equivalence*

$$f_!(\mathcal{F})[-n] \simeq f_*(\mathcal{F}).$$

Proof. By Theorem 3.6 we have the following equivalence

$$f_!(\mathcal{F} \otimes D_p) \simeq f_*(\mathcal{F})$$

but as in this case $D_p \simeq 1[-n]$ we obtain

$$f_!(\mathcal{F} \otimes 1[-n]) \simeq f_*(\mathcal{F})$$

and thus

$$f_!(\mathcal{F})[-n] \simeq f_*(\mathcal{F}).$$

proving our claim.

$\square$

Let us give an example

Example 3.8. For a parallelizable manifold X of dimension d we have that $D_p \simeq 1[-d]$ so we wish to recover classical Poincaré duality in this case. For $E \in D(X)$ by Proposition 3.4 we have

$$p_!(E \otimes D_p) \cong p_*(E)$$

but as in this case $D_p \cong 1[-n]$ we obtain

$$p_!(E \otimes 1[-n]) \cong p_*(E)$$

Then taking π_{-k} on each side we obtain

$$\pi_{-k}(p_!(E \otimes 1[-n])) \cong \pi_{-k}(p_*(E))$$

and as $\pi_{-k}(p_*(E) = H^k(X,E)$ by definition and $\pi_{-k}(p_!(E \otimes 1[-n])) \cong \pi_{-k}(p_!(E))[-n])$ and this is $\pi_{-k}(p_!(E))[-n]) \cong \pi_{n-k}(p_!(E)) \cong H_{n-k}(X;E)$ we have proven our claim.

4. Künneth formula

In this section we recover the Künneth formula using the six functor formalism for anima.

Theorem 4.1. *Let X, Y be anima, and $p_X \colon X \to 1$ and $p_Y \colon Y \to 1$ the unique maps to the final anima, in this situation we have a canonical equivalence*

$$p_!(\mathcal{F} \boxtimes \mathcal{G}) \simeq p_{X!}\mathcal{F} \otimes p_{Y!}\mathcal{G}$$

to be specified in the proof.

Proof.

$$p_!(\mathcal{F} \boxtimes \mathcal{G}) \simeq p_!(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})) \tag{4.2}$$

$$\simeq p_{X!}(p_{1!}(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})) \tag{4.3}$$

$$\simeq p_{X!}(\mathcal{F} \otimes p_{1!}p_2^*\mathcal{G})) \tag{4.4}$$

$$\simeq p_{X!}(\mathcal{F} \otimes p_X^* p_{Y!}\mathcal{G})) \tag{4.5}$$

$$\simeq p_{X!}(\mathcal{F}) \otimes p_{Y!}(\mathcal{G}) \tag{4.6}$$

where we used base change in Equation (4.3) and Equation (4.5) and the projection formula in Equation (4.4) and Equation (4.6), thus proving our claim. □

5. Generalised Mayer–Vietoris

In this section we will recover the well known sequence of Mayer and Vietoris.
We begin by a preliminary lemma.

Lemma 5.1. *The functor* $D = \mathbf{Fun}(-, \mathrm{Sp})\colon \mathbf{An}^{\mathrm{op}} \to \mathbf{LPr}$ *takes colimits to limits.*

Proof. This is immediate from the adjunction description of limits and colimits and that the functor is contravariant so it reverses units and counits, and as limits and colimits are computed pointwise this proves the claim. □

Theorem 5.2. *The functor* $\mathbf{An}^{\mathrm{op}} \to \mathrm{End}(\mathrm{Sp})$ *that assigns* p *to* $p_*p^*\colon \mathrm{Sp} \to \mathrm{Sp}$ *exists and takes colimits in* $\mathbf{An}$ *to limits in* $\mathrm{End}(\mathrm{Sp})$*.*

Proof. First note that $p^* \to p_*p^*$ promotes to a functor, by Proposition 3.2. Suppose that $S\colon K \to \mathbf{An}$ is a colimit diagram and $1\colon K \to \mathbf{An}$ be the finial diagram, then we have a unique natural transformation $p\colon S \to 1$. If we compose S the cohomology functor $\mathbf{An}^{\mathrm{op}} \to \mathrm{End}(\mathrm{Sp})$ we get a diagram $p_*p^*\colon K^{\mathrm{op}} \to \mathbf{An}^{\mathrm{op}} \to \mathrm{End}(\mathrm{Sp})$.
We want to prove that this is a limit diagram.
Since checking equivalence of functors can be checked pointwise it suffices to for $E \in \mathrm{Sp}$ to check that the diagram $p_*p^*(E)\colon (K^{\mathrm{op}}) \to \mathrm{Sp}$ is a limit diagram.
Furthermore we know that $\mathrm{Map}(X,-)$ preserves and reflects colimits by [Lur09, Proposition 5.1.3.1], and thus it suffices to prove that for all $X \in \mathrm{Sp}$ the diagram of mapping anima $\mathrm{Map}(X, p_*p^*E)\colon (K^{op}) \to \mathbf{An}$ is a limit diagram

By adjunction we have

$$\mathrm{Map}(X,p_*p^*E) \simeq \mathrm{Map}(p^*X,p^*E)$$

and thus it suffices to prove it for $\mathrm{Map}(p^*X,p^*E)$.

Since $S\colon K \to \mathbf{An}$ is a colimit diagram we have by Lemma 5.1 that D takes colimits to limits in $\mathbf{LPr}$, and thus $D(S)\colon K^{op} \to \mathbf{LPr}$ is a limit diagram in $\mathbf{LPr}$.

Now we use that limits commute with limits to finish our proof.

Indeed for any $i \in K$ the square

$$\begin{tikzcd}
\mathrm{Map}_{D(S_i)}(p_i^*X,p_i^*E) \arrow[r] \arrow[d] & \mathbf{Fun}(\Delta^1,D(S_i)) \arrow[d] \\
1 \arrow[r] & D(S_i) \times D(S_i)
\end{tikzcd}$$

is Cartesian by definition of Map. By taking limits we obtain

$$\begin{tikzcd}
\lim_i \mathrm{Map}_{D(S_i)}(p_i^*X,p_i^*E) \arrow[r] \arrow[d] & \lim_i \mathbf{Fun}(\Delta^1,D(S_i)) \simeq \mathbf{Fun}(\Delta^1,D(S_\infty)) \arrow[d] \\
1 \arrow[r] & \lim_i(D(S_i) \times D(S_i)) \simeq D(S_\infty) \times D(S_\infty)
\end{tikzcd}$$

is Cartesian (as limits preserve limits) where $\infty \in (K^{op})^{\triangleleft}$ is the limit point.

Thus we obtain that

$$\lim_i \mathrm{Map}_{D(S_i)}(p_i^*X,p_i^*E) \simeq \mathrm{Map}_{D(S_\infty)}(p_\infty^*X,p_\infty^*E)$$

thus proving that $\mathrm{Map}(X,p_*p^*E)$ is a limit diagram, and thus proving the claim. □

Appendix A. Fourier—Mukai transform

Construction A.1. First we let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and $X \in \mathbf{An}$. First we wish to define a functor $\mathbf{Fun}(\mathcal{C},\mathcal{D}) \to \mathbf{Fun}(\mathcal{C}^X,\mathcal{D}^X)$. By the $\times D \dashv \mathbf{Fun}(D,-)$ adjunction we have that it is equivalent to provide a map

$$\mathcal{C}^X \times \mathbf{Fun}(\mathcal{C},\mathcal{D}) \to \mathcal{D}^X$$

and here we may take $(G,F) \mapsto F \circ G$. We call this map $(-)^X$.

Remark A.2. This is explicitly the map

$$\mathbf{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{\eta} \mathbf{Fun}(\mathcal{C}^X,\mathcal{C}^X \times \mathbf{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{\circ} \mathbf{Fun}(\mathcal{C}^X,\mathcal{D}^X)$$

by the unit counit description of adjoints.

Definition A.3. In the context of Construction A.1 choose $\mathcal{C} = \mathrm{Sp}^X$ and by the equivalence $\mathbf{Fun}(X, \mathrm{Sp}^X) \simeq \mathbf{Fun}(X \times X, \mathrm{Sp})$ we obtain a map

$$(-)^X : \mathbf{Fun}(\mathrm{Sp}^X, \mathrm{Sp}) \to \mathbf{Fun}(\mathrm{Sp}^{X \times X}, \mathrm{Sp}^X)$$

Sending $F \mapsto F^X$ and then evaluating F^X at $\Delta_! 1_X \in \mathbf{Fun}(X \times X, \mathrm{Sp})$ is what we call the functor *induced by evaluating at* $\Delta_! 1_X \in \mathbf{Fun}(X \times X, \mathrm{Sp})$.

Proposition A.4. *For* $X, Y \in \mathbf{An}$ *the functor from* $\mathrm{Sp}^{X \times Y}$ *to* $\mathbf{Fun}^L(\mathrm{Sp}^X, \mathrm{Sp}^Y)$ *given by*

$$D \mapsto \mathrm{pr}_{Y!}(\mathrm{pr}_X^*(-) \otimes D) \colon \mathrm{Sp}^X \to \mathrm{Sp}^Y,$$

where $\mathrm{pr}_X \colon X \times Y \to X$ *and* $\mathrm{pr}_Y \colon X \times Y \to Y$ *are the projections. is an equivalence. It has inverse given by evaluation at* $\Delta_! 1_X \in \mathbf{Fun}(X \times X, \mathrm{Sp})$.

Proof. This is [Cno23, Theorem 2.32] applied to $\mathcal{B} = \mathbf{An}$ and $\mathcal{C} = \mathrm{Sp}$, and thus proving our claim. □

Corollary A.5. *Let* $X \in \mathbf{An}$ *and let* F, G *be functors from* $(\mathrm{Sp}^X \to \mathrm{Sp}^Y)$ *then evaluation at* $\Delta_! 1_X$ *induces an equivalence*

$$\mathrm{Map}(F,G) \simeq \mathrm{Map}(F^X(\Delta_! 1_X), G^X(\Delta_! 1_X))$$

Proof. This follows as these are the mapping spaces of the equivalence in Proposition A.4, and thus we have proven our claim. □

Appendix B. The proof of Proposition 3.2

We thank Maxime Ramzi for explaining the proof below. Let $q \colon (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}//} \to \widehat{\mathbf{Cat}}_\infty$ denote the coCartesian straightening of $\mathbf{Fun}(\mathcal{C},-) \colon \widehat{\mathbf{Cat}}_\infty \to \widehat{\mathbf{Cat}}_\infty$

Proposition B.1. *There is a map of coCartesian fibrations over* $(\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/}$ *of the form*

$$(\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/} \times \mathbf{Fun}(\mathcal{C},\mathcal{C}) \to (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/} \times_{(\widehat{\mathbf{Cat}}_\infty)} (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}//}$$

which on fibers over (D,f) *is given by the functor* $\mathbf{Fun}(\mathcal{C},\mathcal{C}) \xrightarrow{f \circ -} \mathbf{Fun}(\mathcal{C},\mathcal{D})$.

Proof. We have a natural map $\mathrm{Map}(\mathcal{C},-) \times \mathbf{Fun}(\mathcal{C},\mathcal{C}) \to \mathbf{Fun}(\mathcal{C},-)$ and thus by [Lur09, Theorem 3.3.10] we obtain a map of coCartesian fibrations

$$(\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/} \times \mathbf{Fun}(\mathcal{C},\mathcal{C}) \to (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}//}$$

over $(\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/}$, and by universal property of pullbacks and by [Lur09, Proposition 2.4.1.3] we obtain the desired map of coCartesian fibrations over $(\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/}$, and the second claim is by construction, thus proving our claim. □

Remark B.2. If we restrict $(\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/}$ to $(\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/}$ then we get a map of coCartesian fibrations

$$(\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/} \times \mathbf{Fun}(\mathcal{C},\mathcal{C}) \to (\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/} \times_{(\widehat{\mathbf{Cat}}_\infty} (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}//},$$

which fiberwise is a left adjoint.

Corollary B.3. *The functor*

$$(\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/} \times \mathbf{Fun}(\mathcal{C},\mathcal{C}) \to (\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/} \times_{(\widehat{\mathbf{Cat}}_\infty} (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}//}$$

admits a right adjoint G with respect to $(\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}//}$. In particular it admits a right adjoint. Furthermore on fibers over (D,f) G is given by the right adjoint to the functor $\mathbf{Fun}(\mathcal{C},\mathcal{C}) \xrightarrow{f\circ -} \mathbf{Fun}(\mathcal{C},\mathcal{D})$.

Proof. The first part follows by Proposition B.1 combined with [Lur17, Proposition 7.3.2.6], and the second part follows by [Lur17, Proposition 7.3.2.5], and thus proving the claim. □

Corollary B.4. *The composition*

$$(\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/} \to (\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/} \times_{(\widehat{\mathbf{Cat}}_\infty)} (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/} \to$$

$$(\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/} \times_{(\widehat{\mathbf{Cat}}_\infty)} (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}//} \xrightarrow{G} (\widehat{\mathbf{Cat}}_\infty)^L_{\mathcal{C}/} \times \mathbf{Fun}(\mathcal{C},\mathcal{C}) \to \mathbf{Fun}(\mathcal{C},\mathcal{C})$$

where the first functor is the diagonal functor, the second is induced by the inclusion $(\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}/} \to (\widehat{\mathbf{Cat}}_\infty)_{\mathcal{C}//}$ using the natural map $\mathrm{Map}(\mathcal{C},-) \to \mathbf{Fun}(\mathcal{C})$, and the last is the projection map. This composition is given by $f \mapsto f^R f$ on objects, where f^R denotes the right adjoint of f.

Proof. The functor sends on objects $f \mapsto (f,f) \mapsto (f,f) \mapsto (f^R f, f^R, f) \mapsto f^R f$, and thus proving the claim. □

Proof of Proposition 3.2. This follows directly from Corollary B.4. □

Appendix C. f-proper and the 2-category of kernels

In this subsection we assume that $\mathcal{C}$ is an ∞-category with a final obejct S, which admits finite limits and that is a six functor formalism $D\colon \mathbf{Corr}(\mathcal{C}) \to \mathbf{LPr}_{st}$, in the sense of [Mann22, Definition A.5.7.]. First we define the

Definition C.1. The 2-category of kernels $\mathbf{LZ}_{D,S}$ is the following 2-category.

1) The objects $\mathbf{LZ}_{D,S}$ are the objects of $\mathcal{C}$.

2) For every two objects $X,Y \in \mathrm{ob}(\mathbf{LZ}_D)$ then we have $\mathrm{Map}_{\mathbf{LZ}_{D,S}}(X,Y) = D(X \times_S Y)$

3) For every triple $X_1,X_2,X_3 \in \mathrm{ob}(\mathbf{LZ}_{D,S})$ the composite functor

$$\mathrm{Map}_{\mathbf{LZ}_{D,S}}(X_2,X_3) \times \mathrm{Map}_{\mathbf{LZ}_{D,S}}(X_1,X_2) \to \mathrm{Map}_{\mathbf{LZ}_{D,S}}(X_1,X_3)$$

is defined as $(A,B) \mapsto (p_{1,3})_!(p_{1,2}^*(B) \otimes p_{2,3}^*(A))$ where $p_{i,j}\colon X_1 \times X_2 \times X_3 \to X_i \times X_j$ are projection maps.

4) For every object $X \in \mathrm{ob}(\mathbf{LZ}_{D,S})$ the identity 1-map is $\mathrm{id}_X = \Delta_!(1_X)$ where $\Delta\colon X \to X \times_S X$ is the diagonal map.

Remark C.2. The proof that this is a 2-category is given in [Sch22].

Definition C.3. Given a map $f\colon X \to S$, $A \in \mathcal{D}(X)$ is *f-proper* if when viewed as element of $D(X) \simeq D(S \times_S X) = \mathrm{Hom}_{\mathbf{LZ}_D}(S,X)$ i.e. a 1-map it admits a right adjoint.

Definition C.4. We define the *norm map* to be the mate under the $f^* \dashv f_*$ adjunction defined by

$$\begin{aligned} f^* f_!(- \otimes p_{2*}\,\mathrm{map}(p_1^* A, \Delta_!(1_X)))) &\simeq p_{1!}(p_2^*(-) \otimes p_2^* p_{2*}\,\mathrm{map}(p_1^* A,\Delta_!(1_X))) \\ &\to p_{1!}(p_2^*(-) \otimes \mathrm{map}(p_1^* A,\Delta_!(1_X)))) \\ &\to \mathrm{map}(A,-). \end{aligned}$$

Proposition C.5. *Let $A \in D(X)$, then A is f-proper if and only if the norm map*

$$f_!(- \otimes p_{2*}\,\mathrm{map}(p_{1*}(A),\Delta_!(1_X))) \to f_*\,\mathrm{map}(A,-)$$

is an equivalence.

Proof. This is [Sch22, Proposition 6.9][2]. □

[2]The proof that this is indeed the norm map is not given in Scholze's notes, but will be in an upcoming paper by Heyer and Mann.

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